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Percolation on the average and spontaneous magnetization for q-states Potts model on graph

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Abstract

We prove that the q-states Potts model on graph is spontaneously magnetized at finite temperature if and only if the graph presents percolation on the average. Percolation on the average is a combinatorial problem defined by averaging over all the sites of the graph the probability of belonging to a cluster of a given size. In this paper, we obtain an inequality between this average probability and the average magnetization, which is a typical extensive function describing the thermodynamic behaviour of the model.

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1. Introduction

The interplay between spin models and percolation has been put into evidence since the fundamental work of Fortuin and Kasteleyn [1], where it is shown that percolation is the limit for $q \to 1$ of the random cluster representation of q-states Potts model. This representation, as already pointed out in [1], is very general and can be performed on any discrete structure, i.e. on graph.

A further step has been the result proved in [2], showing that, on lattices, the existence of percolation implies Potts transition for any value of q. The main purpose of this work is the extension of the result to generic networks.

An important issue in this direction has been a mathematical paper [3] proving that Potts model has more than one Gibbs measure if it is defined on graphs showing percolation. The multiplicity of Gibbs measures is the most used definition of symmetry breaking in mathematical literature. However, it does not always correspond to the thermodynamic concept of phase transition.

Indeed, in the study of the thermodynamic properties of physical systems, we are interested in the free energy and its derivatives such as the average magnetization, the susceptibility,

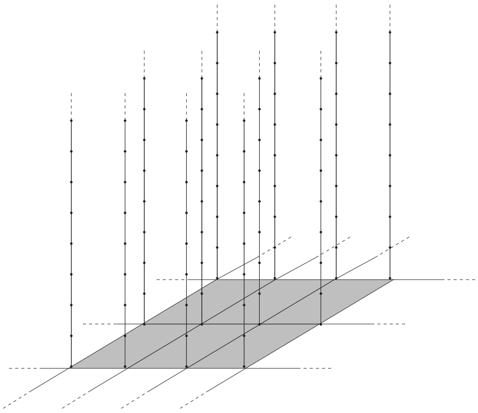

Figure 1. The brush graph is obtained by adding a linear chain to each site of a two-dimensional lattice.

the specific heat, which are typical average extensive quantities. On the other hand, the existence of a single Gibbs measure, as explained in [3], depends on a local parameter, that is the magnetization of a site of the graph. On lattices, translation invariance implies that local and average quantities coincide (the proof of [2] depends on this fact). However, for inhomogeneous structures the results can be different in the two cases. For example, on the brush graph (see figure 1) as we will prove in detail, there is not a single Gibbs measure, but the thermodynamic functions do not present discontinuities and the spontaneous magnetization is zero.

A generalization of the result [2], which takes into account the thermodynamic behaviour of the system, requires the introduction of the new problem of percolation on the average. A graph G is said to present percolation on the average if the average, over all sites, of the probability of belonging to a cluster of size greater than l remains positive when $l \to \infty$. We will prove that the q-states Potts model on a graph G is spontaneously magnetized if and only if G presents percolation on the average.

Therefore, the study of spin models with discrete symmetry leads to a classification of graph topology which can be formulated in terms of a purely combinatorial problem.

An analogous result [4] has been previously obtained for classical spin models with continuous symmetry, where the thermodynamic behaviour depends on a different combinatorial problem, i.e. random walks. In particular, the models present spontaneous magnetization if and only if the graph is transient on the average [5] (i.e. a graph where the average probability for a random-walk of ever returning to the starting site is smaller than one).

These two classifications are different, indeed, it is well known that lattices presenting percolation (i.e. lattices with dimension $d \geqslant 2$) are transient on the average only if $d \geqslant 3$. In recent works [6], it has been proved that the 'anomalous' behaviour of presenting average percolation in a recurrent on the average structure is typical not only of two-dimensional lattices, but also of low-dimensional networks, such as the Sierpinski carpet. Therefore, a general result clarifying the relation between the two classifications is still lacking. Furthermore, up to now we do not know if percolation on the average is related to graph topology by a simple parameter, as it is the spectral dimension [7] in the case of transience on the average.

This paper is organized as follows. In the next section we introduce the Potts model and percolation on graph, we define percolation on the average and we state the main result of the paper, i.e. Potts model is spontaneously magnetized if and only if it is defined on a graph presenting percolation on the average. In the last two sections we prove the theorem.

2. The q-states Potts model and percolation on graph

A graph G (see e.g. [8] for an introduction to graph theory) is a countable set V of vertices or sites i connected pairwise by a set E of unoriented edges or links $(i, j) = (j, i)$. Connected sites are called nearest neighbours and we denote with z_i the connectivity (number of neighbours) of the site i. A path is a sequence of consecutive links $\{(i, k)(k, h) \dots (n, m)(m, j)\}$. The chemical distance $r_{i,j}$ is the length (number of links) of the shortest path connecting the sites i and j. The Van Hove sphere $S_{o,r}$ of centre o and radius r is the set of the sites of G such that $r_{o,i} \leqslant r$. We will call $E_{o,r}$ the set of all $(i, j) \in E$ such that $i, j \in S_{o,r}$, ∂S_r the set of the vertices of S_r connected with the rest of the graph and we denote with $|\cdot|$ the cardinality of a set. Let ϕ_i be a real function of the sites of an infinite graph, the average in the thermodynamic limit $\bar{\phi}$ is

$$\bar{\phi} \equiv \lim_{r\to\infty} \frac{\sum_{i\in S_{o,r}} \phi_i}{|S_{o,r}|}. \tag{1}$$

Interesting properties of the thermodynamic average, such as the independence of the choice of the centre o, are proved in [5]. Therefore, in the following we will drop the index o. The measure $\|S\|$ of a subset S of V is the average value $\overline{\chi(S)}$ of its characteristic function $\chi_i(S)$ defined by $\chi_i(S) = 1$ if $i \in S$ and $\chi_i(S) = 0$ if $i \notin S$.

Important constraints on graph topology follow from the requirement to describe a physical network. Real systems, indeed, have bounded local energy and they are embedded in a finite-dimensional space. Therefore, we will consider only connected graphs with bounded connectivity ($z_i < z_{\max}\ \forall i \in E$) and such that

$$\lim_{r\to\infty} \frac{|\partial S_r|}{|S_r|} = 0. \tag{2}$$

The definition of the q-states Potts model (see e.g. [9] for a review) on an infinite graph G first requires to introduce the model on the Van Hove spheres S_r. For each site $i \in S_r s_i$ is a q-state function $s_i = 1 \dots q$ and the Hamiltonian is given by:

$$H_r = \sum_{(i,j)\in E_r} (1 - \delta(s_i, s_j)) + h \sum_{i\in S_r} (1 - \delta(s_i, 1)) \tag{3}$$

where the Kronecker delta-function is $\delta(s_i, s_j) = 1$ if $s_i = s_j$ and $\delta(s_i, s_j) = 0$ otherwise. Note that the case $q = 2$ is the Ising model. In the canonical ensemble the partition function Z_r is given by the sum of the Boltzmann weight $\exp(-\beta H_r)$ over all possible configurations $\{s_i\}$; β represents the inverse temperature of the system. Thermodynamic properties of statistical models are described by extensive order parameters, in this case the average magnetization:

$$\begin{aligned} M_r(\beta, q, h) &= |S_r|^{-1} \sum_{i\in S_r} \langle \delta(s_i, 1) - q^{-1} \rangle_r \\ &\equiv |S_r|^{-1} \sum_{i\in S_r} Z_r^{-1} \sum_{s_1 \dots s_{S_r}} (\delta(s_i, 1) - q^{-1})\, \mathrm{e}^{-\beta H_r} \end{aligned} \tag{4}$$

where $\langle\cdot\rangle_r$ denotes the thermal average. The Potts model on G presents spontaneous magnetization if there exists β_c such that for all $\beta > \beta_c$

$$\lim_{h\to 0} M(\beta, q, h) \equiv \lim_{h\to 0}\lim_{r\to\infty} M_r(\beta, q, h) > 0. \tag{5}$$

The existence of the thermodynamic limit $r \to \infty$ is always assumed in this paper.

Percolation (see e.g. [10, 11] for a mathematical and a physical treatment) is defined by introducing a probability $p, 0 \leqslant p \leqslant 1$ and each link $(i, j) \in E$ is declared to be open with probability p and closed with probability $1 - p$ independently. The cluster C_i containing the site i is the set of all the sites connected with i by a path of open links. The i-size of the cluster C_i is the maximum chemical distance from i of the sites of C_i. We call $P_i(l, p)$ the probability for i to belong to a cluster of i-size $\geqslant l$. A graph G is said to present (local) percolation if there exists a probability $p < 1$ such that $\lim_{l\to\infty} P_i(l, p) > 0$. On a graph with bounded connectivity this property is independent of i [10].

A graph presents percolation on the average if there exists a probability p such that

$$\lim_{l\to\infty} \overline{P(l, p)} \equiv \lim_{l\to\infty}\lim_{r\to\infty} |S_r|^{-1} \sum_{i\in S_r} P_i(l, p) > 0. \tag{6}$$

Percolation on the average is a new combinatorial problem and it gives rise to a graph classification which turns out to be fundamental for understanding the thermodynamic behaviour of physical models on graph. In the main theorem of the paper, indeed, we prove that the q-states Potts model is spontaneously magnetized if and only if it is defined on a graph presenting percolation on the average, obtaining a generalization to inhomogeneous structures of the classical result for lattices by Aizenman *et al* [2]. The proof is mainly inspired by [1, 3].

First, following [1], we introduce a representation of the q-states Potts model in terms of the q-random cluster model on the supplemented graph G'. Then, from an important property of the random cluster stated in [3], we get an inequality between the local magnetization and the percolation probability on the supplemented graph: this is an extension of [3] to the case $h \neq 0$. In the second part of the proof we take the thermodynamic average and we show that the physical requirements on the graph allow to formulate the inequality for the average magnetization in terms of average percolation on G. The theorem directly follows from this inequality.

Let us first show that local and average percolation define different classifications of inhomogeneous networks. In particular, the brush graph (figure 1) presents local percolation but no percolation on the average. If we choose p larger than the threshold of the two-dimensional percolation and we call r the distance of i from the plane of the brush, we have that $P_i(l, p) > P_{sq}(l - r, p)p^r$ ($P_{sq}(l, p)$ is the probability that a site of a two-dimensional lattice belongs to a cluster of size $>l$) and then $\lim_{l\to\infty} P_i(l, p) > p^r \lim_{l\to\infty} P_{sq}(l-r, p) > 0$. On the other hand, for all the sites of the brush at a distance from the plane greater than l we have $P_i(l, p) = 2p^l - p^{2l}$, $2p^l - p^{2l}$ is the probability for a site of a linear chain to belong to a cluster of i-size greater than l. We call R_l the subset of the brush graph given by the sites whose distance from the plane is smaller than l. We have $\|R_l\| = 0$ and

$$\lim_{l\to\infty} \overline{P(l, p)} \leqslant \lim_{l\to\infty} \|R_l\| + (1 - \|R_l\|)(2p^l - p^{2l}) = 0. \tag{7}$$

Therefore, on the brush graph the q-states Potts model does not have a single Gibbs measure [3]; however the system is not spontaneously magnetized and the thermodynamic functions are analytical.

3. Random cluster representation of the Potts model

Let us define the supplemented q-random cluster model [1]. We call G' the graph obtained by adding to G a supplementary site o connected by a new link to each site of G. The supplemented spheres are obtained by adding to the spheres S_r the new vertex o and the relevant edges. We will call S_r' and E_r' the set of sites and the set of links of the supplemented sphere. The configurations ξ of the supplemented random cluster model on S_r' are obtained by declaring each link of E_r' open or closed. The supplemented q-random cluster model [1] is defined by choosing each configuration according to the probability distribution:

$$\mu(p, p_o, q, \xi) = p^{N_S(\xi)}(1-p)^{|E_r|-N_S(\xi)} p_o^{N_o(\xi)}(1-p_o)^{|S_r|-N_o(\xi)} q^{C(\xi)} \tag{8}$$

where p, p_o and q are real parameters of the model ($q \geqslant 1, 0 \leqslant p \leqslant 1$ and $0 \leqslant p_o \leqslant 1$). $N_S(\xi)$ is the number of open links in E_r, $N_o(\xi)$ the number of open links connecting o to S_r and $C(\xi)$ the number of clusters in the configuration ($|S_r|$ and $|E_r|$ are respectively the cardinalities of S_r and E_r). For $q = 1$ and $p_o = 0$ the random cluster model is exactly the percolation model restricted to the sphere S_r.

As already pointed out in the original work by Fortuin and Kasteleyn [1], q-states Potts model and q-random cluster model are equivalent on any graph G. Indeed if we fix $p = 1-\mathrm{e}^{-\beta}$ and $p_o = 1 - \mathrm{e}^{-\beta h}$, their partition functions coincide and

$$\langle \delta(s_i, 1) - q^{-1} \rangle_r = P_i^o(p, p_o, q) \equiv \int F_i^o(\xi)\mu(p, p_o, q, \xi) \tag{9}$$

where $F_i^o(\xi) = 1$ if $o \in C_i$ and $F_i^o(\xi) = 0$ otherwise. Then, $P_i^o(p, p_o, q)$ represents the probability for the site i to belong to the same cluster of the supplemented site o.

The space of configurations of the random-cluster model is equipped by the partial order: $\xi \preceq \eta$ if the set of open links in ξ is included or equal to the set of open links in μ. We will say that the probability distribution $\mu(\xi)$ stochastically dominates the probability distribution $\mu'(\xi)$ ($\mu' \preceq_D \mu$) if:

$$\int f \, \mathrm{d}\mu' \leqslant \int f \, \mathrm{d}\mu \tag{10}$$

for all increasing function f.

In [3], it is proved that important inequalities for the probability distributions follow from FKG [12] theorem. These inequalities can be easily generalized to the case of supplemented graph, obtaining

$$\mu(p', p_o', 1, \xi) \preceq_D \mu(p, p_o, q, \xi) \preceq_D \mu(p, p_o, 1, \xi) \tag{11}$$

where $p' = p(q - p(q-1))^{-1}$ and $p_o' = p_o(q - p_o(q-1))^{-1}$.

Since $F_i^o(\xi)$ is an increasing function, from (9) and (11) we have

$$P_i^o(p', p_o', 1) \leqslant \langle \delta(s_i, 1) - q^{-1} \rangle_r \leqslant P_i^o(p, p_o, 1) \tag{12}$$

(12) is an extension to the case $h \neq 0$ of the inequality for local magnetization proved in [3].

4. Spontaneous magnetization and percolation on the average

Let us first take the average of (12) over all the sites of the sphere S_r. We obtain

$$|S_r|^{-1} \sum_{i \in S_r} P_i^o(p', p_o', 1) \leqslant M_r(\beta, q, h) \leqslant |S_r|^{-1} \sum_{i \in S_r} P_i^o(p, p_o, 1). \tag{13}$$

Inequalities (13) provide upper and lower bounds for the magnetization of the Potts model in terms of the percolation model defined on the supplemented sphere S'_r. Let us reformulate these bounds in terms of the percolation on G.

First we prove a property of graphs with bounded connectivity and satisfying (2). In the probability measure defined on S_r by (8) with $q = 1$ and $p_o = 0$, we call $P_{i,r}(l, p)$ the probability for the site i to belong to a cluster of i-size greater than l. Let us show that in the thermodynamic averages one can substitute $P_{i,r}(l, p)$ with the probability defined on the infinite graph $P_i(l, p)$. We will call $S^+_{l,r}$ the subset of S_r of all the sites i such that the distance of i from the border ∂S_r is larger than l and $S^-_{l,r}$ its complementary in S_r. For the sites $i \in S^+_{l,r}$ all the clusters C_i of i-size smaller than l are included in S_r, then the probability of belonging (or not belonging) to one of these clusters is equal in percolation on G and in percolation on S_r. Furthermore we have that

$$\|S^-_{l,r}\| = \lim_{r\to\infty} |S_r|^{-1}|S^-_{l,r}| \leqslant \lim_{r\to\infty} |S_r|^{-1}|\partial S_r| z^{l+1}_{\max} = 0 \tag{14}$$

where we used (2) and the boundedness of the coordination number. Then $P_{i,r}(l, p)$ and $P_i(l, p)$ differ only on a set of zero measure and in the thermodynamic averages we can exchange one for the other.

Let us now prove the main theorem of the paper. From the boundedness of the correlation number and the independence of percolation probabilities, one obtains the inequality

$$P^o_i(p, p_o, 1) \leqslant \left(1 - (1 - p_o)^{z^l_{\max}}\right)(1 - P_{i,r}(l, p)) + P_{i,r}(l, p). \tag{15}$$

Indeed $\left(z^l_{\max} - 1\right)/(z_{\max} + 1)$ is the maximum number of sites in a cluster of i-size $<l$ and then $\left(1 - (1 - p_o)^{z^l_{\max}}\right)$ is an upper bound on the probability that a site of the cluster is connected to o. From (13) and (15) taking the thermodynamic limit $r \to \infty$ we have

$$M(\beta, q, h) \leqslant \lim_{r\to\infty} \frac{1}{|S_r|} \sum_{i\in S_r} \left(\left(1 - (1 - p_o)^{z^l_{\max}}\right)(1 - P_i(l, p)) + P_i(l, p)\right). \tag{16}$$

Letting $h \to 0$, since also $p_o \to 0$, we obtain

$$\lim_{h\to 0} M(\beta, q, h) \leqslant \lim_{h\to 0} \left(1 - (1 - p_o)^{z^l_{\max}}\right) + \overline{P(l, p)} = \overline{P(l, p)}. \tag{17}$$

Now we have to consider the first inequality in (13). From the probability independence in percolation, we get

$$P^o_i(p', p'_o, 1) \geqslant (1 - (1 - p'_o)^{l+1}) P_{i,r}(l, p'). \tag{18}$$

Here $l + 1$ is the minimum number of sites in a cluster of i-size $\geqslant l$, and then $(1 - (1 - p'_o)^{l+1})$ is a lower bound on the probability that a site of the cluster is connected to o. Taking the thermodynamic average we have

$$M(\beta, q, h) \geqslant (1 - (1 - p'_o)^{l+1})\overline{P(l, p')}. \tag{19}$$

Letting first $l \to \infty$ and then $h \to 0$ we have

$$\lim_{h\to 0} M(\beta, q, h) \geqslant \lim_{l\to\infty} \overline{P(l, p')}. \tag{20}$$

Finally, from (17) and (20) one has

$$\lim_{l\to\infty} \overline{P(l, p')} \leqslant \lim_{h\to 0} M(\beta, q, h) \leqslant \lim_{l\to\infty} \overline{P(l, p)}. \tag{21}$$

If the graph presents percolation on the average, there exists a probability p' such that $\lim_{l\to\infty} \overline{P(l, p')} > 0$ and then the first inequality in (21) implies that the model for $\beta = \log(1 + p'q(1 - p')^{-1})$ is magnetized. On the other hand, if $\lim_{l\to\infty} \overline{P(l, p)} = 0$ for all values of p, the second inequality in (21) implies that the magnetization of the system is zero for all temperatures and this concludes the proof.

The result can be easily generalized to Potts models with disordered ferromagnetic couplings described by the Hamiltonian $H_r = \sum_{(i,j)\in E_r} J_{ij}(1 - \delta(s_i, s_j)) + h \sum_{i\in S_r} (1 - \delta(s_i, 1))$ with $0 < \epsilon < J_{ij} < K$ for all $(i, j) \in G$.

On the other hand, an important open question is the possibility of extending the proof to different ferromagnetic spin models with discrete symmetry, such as the clock model. As mentioned in the introduction, another basic problem is a better understanding of the relation between percolation on the average, transience on the average and graph topology.

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